Stochastic Model of Thin Market with an Indivisible Commodity[#]

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According to the usual definition, a market is said to be *thin* if the activity of its participants is low. A market with houses in a small town, a market with antiquities or a market with the stocks of a small firm could serve as examples.

Since there are different trading mechanisms established in individual markets, and since these differences may not be neglected, each type of (thin) market requires a special treatment. Hence, we may divide the existing works into those describing auctions (cf. the bibliography by Klemperer, 1999), those devoted to labor market (cf, Roth and Xing, 1994, Crawford, 1991, or Coles and Smith, 1998, for instance) and those describing thin markets with securities (cf. Lo and MacKinlay, 1990, for instance).

The present paper is devoted to organized (thin) markets in which the market price is determined (by a market organizer) so that the total trading volume is maximized. In particular, the price is equal to the average of the prices maximizing the traded volume. For the purposes of the present work, we shall call this method of price determination *organized auction*.[1]

As it was already indicated, it does not suffice to model thin markets by (deterministic) demand and supply curves, for, in thin markets, we can observe fluctuations that are not caused by an arrival of new information: the most common "fluctuation" of this type is that, sometimes, no transaction is made at the market. Another issue, calling for a special care, is the indivisibility of the traded commodity: neither it may be neglected.

The present paper reflects all the problems mentioned by the previous paragraph: The markets' fluctuations are modeled by stochastic demand and supply curves, the situation of "no trade" is taken into account[2] and the individual demand and supply curves are assumed to be jump piecewise constant.

The paper presents two different models – the "finite participants" model with a finite number of buyers and sellers, and the "continuous" model with continuous expected demand and supply curves. For both models, formulas of the joint distribution of the market price P and the traded volume Q are introduced.

In the "finite participants" model, it is assumed that the number of sellers and the number of buyers are finite and that the amounts, offered by sellers, and the amounts, demanded by buyers, are Poisson distributed random variables. According to the author of

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[1]   This method of the price determination is used in RM-system (one of the Czech stock markets), for instance.

[2]   In our model, the latter situation is modeled by adding an extra element to the set of possible values of the market price.

the paper, the usage of this distribution is quite natural: similarly to various models of queuing systems, the Poisson distribution of the amounts is a consequence of a quite acceptable assumption that the appearance of the agents' offers (demands) during the time between the auctions is driven by the Poisson process.[3] The formula of the distribution of (P,Q) in this model has the form of a finite sum.

The "continuous" model is similar to the "finite participants" one. The difference lies in the assumption that, despite the "finite participants" model, the jumps of the individual demand and supply curves may realize themselves in an infinite number of points of the real line and that the probability that they realize in a single point is zero. The formula of the joint distribution of (P,Q) has the form of a Lesbegue integral.

The paper is organized as follows – in Section 1, a general formula for the distribution of the price and volume is derived, in Section 2, the "finite participants" model is defined and the distribution of the price and the volume is derived, in Section 3, the same is done for the "continuous" model. Section 4 concludes the paper.

1. Stochastic Model of the Organized Auction

As it was already premised, we assume that the individual demand functions and the individual supply functions are random piecewise constant with integer values. Quite naturally, it is supposed that each demand function is left continuous with the zero limit in infinity, and that each supply function is right continuous with the zero limit in minus infinity. Moreover, we assume that the number of buyers with a non-zero demand function and the number of sellers with a non-zero supply function is finite almost sure.[4]

Given these assumptions, each realization of the aggregate demand function D is piecewise constant left continuous with integer values and with the zero limit in infinity, and each realization of the aggregate supply function S is piecewise constant right continuous with integer values and with the zero limit in minus infinity.

As it was already mentioned, we assume that the market price P is determined as the average price maximizing the total traded volume Q, i.e.

$$P = \begin{cases} (\max M - \min M)/2 & \text{if } D(-\dot\infty) > 0 \text{ or } S(\infty) > 0 \\ undefined & \text{otherwise} \end{cases} \tag{1}$$

where M $=$ $\text{argmax}_{p \in R}\{D(p), S(p)\}$
$D(\text{-}\infty)$ $=$ $D(\text{-}\dot\infty)$ by definition
$S(\infty)$ $=$ $S(\ddot\infty)$ by definition
and

$$Q = \max_{p \in R}\{D(p), S(p)\}. \tag{2}$$

It is easy to prove that, for each positive integer q,

$(Q=q)$ and $(P<p)$ $\Leftrightarrow (X_{[q]} \geq Y_{(q)})$ and $(X_{[q+1]} < Y_{(q+1)})$ and $((X_{[q]}+Y_{(q)})/2 < p)$

$\Leftrightarrow (Y_{(q)} X_{[q]} < 2p - Y_{(q)})$ and $(X_{[q+1]} < Y_{(q+1)})$ $\tag{3}$

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3 If we wanted to make a more intuitive assumption, namely that the i-th agent demands a single unit of the commodity and he comes to trade with a small probability p_i, the Poisson distribution of the amounts would be a good approximation since the alternative distribution with a small p_i is close to the Poisson distribution with the parameter p_i.

4 It will be clear that both the "finite participants" and the "continuous" models satisfy this assumption.

where $\quad X_{[q]} = x \quad (D(x) \geq q)$ and $(D(x^+) < q)$
$\qquad\quad Y_{(q)} = y \Leftrightarrow \quad (S(y) \geq q)$ and $(S(y^-) < q)$

Using the reformulation (3), the formula for the joint distribution of *(P,Q)* may easily be derived:

Theorem 1. *For each positive integer q and each p∈R,*

$$P = \{Q = q, P < p\} = \int_{\{x_q \geq y_q, x_{q+1} < y_{q+1}, x_q + y_q < 2\,p\}} d\mu_q(x_q, x_{q+1}, y_q, y_{q+1}), \qquad (4)$$

$$P = \{Q = 0\} = P\{P = undefined\} = P\{Q = 0, P = undefined\} = \int_{\{x < y\}} d\nu_1(x, y) \qquad (5)$$

where μ_q is the probability distribution of the extended[5] random vector $(X_{[q]}, X_{[q+1]}, Y_{(q)}, Y_{(q+1)})$ and ν_1 is the probability distribution of the extended random vector $(X_{[1]}, Y_{(1)})$.

Proof. *The formula* (4) *follows from the previous text. To prove* (5), *it suffices to notice that $Q = 0 \Leftrightarrow P = undefined \Leftrightarrow (Q = 0$ and $P = undefined) \Leftrightarrow X_{[1]} < Y_{(1)}$.*

2. A Model with a Finite Number of Participants

Let us assume, until the end of the Section 2, that there are *m* buyers and *n* sellers participating in the market. Further, let us suppose that each individual demand function has at most one jump and that the magnitude of the jump is a Poisson random variable. Speaking more exactly, we assume that there exist $x_1, x_2, \ldots, x_m$ and $p_1, p_2, \ldots, p_m$ such that the *i*-th individual demand function is determined by the formula $d_i(x) = I_{\{p \leq x_i\}}(x)D_i$ where I is the indicator function and D_i is a Poisson random variable with an intensity p_i. Analogously, we assume the existence of $y_1, y_2, \ldots, y_n$ and $q_1, q_2, \ldots, q_n$ such that the *j*-th individual supply function is determined by $d_j(y) = I_{\{p \leq y_j\}}(y)S_j$ where S_j is a Poisson random variable with the intensity q_j. Finally, let us assume all the variables D_i and S_j to be mutually independent. Given our assumptions, the aggregate functions are

$$D(p) = \sum_{x_i \geq p} D_i, \quad S(p) = \sum_{y_i \leq p} S_j. \qquad (6)$$

Thanks to the fact that the distribution of the sum of independent Poisson random variables is Poisson with its the parameter equating the sum of the summed variables' parameters,

$$D(x) \sim \mathrm{Po}(\delta(x)), \quad \delta(x) = \sum_{x_i \geq x} p_i \qquad (7)$$

for each $-\infty \leq x \leq \infty$ and

$$S(y) \sim \mathrm{Po}(\sigma(y)), \quad \sigma(y) = \sum_{y_i \leq y} q_j \qquad (8)$$

for each $-\infty < y < \infty$ Hence, we are getting that

$$ED(x) = \delta(x), \quad ES(y) = \sigma(y) \qquad (9)$$

for each $-\infty \leq x \leq \infty$ and $-\infty < y < \infty$.

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5 According to the usual definition, extended random variable is random element that may take values from the set R∪{∞}, see Štěpán (1987) for information about extended random variables. It is clear that the work with random elements taking values in R∪{-∞} is completely analogous.

The following Theorem provides a little complicated but computable formula for the joint distribution of (P,Q):

Theorem 2. *Denote* $x_1, x_2, \ldots, x_m$ *the jumps of* $\delta(\bullet)$*, put* $x_0 = -\infty$*, denote* $y_1, y_2, \ldots, y_n$ *the jumps of* $\sigma(\bullet)$ *and put* $y_{n+1} = \infty$*. If* $q > 0$ *then it holds that*

$$P = \{Q = q, P < p\} = \sum_{i_1=0}^{m'} \sum_{i_2=0}^{m'} \sum_{j_1=1}^{n'+1} \sum_{j_2=1}^{n'+1} I_{i1,i2,j1,j2}(p,q)\pi_2(q, x'_{i_1}, x'_{i_2})\rho_2(q, y'_{j_1}, y'_{j_2}) \qquad (10)$$

$$P = \{Q = 0\} = \sum_{\{x'_i < y'_j : i=0,1,\ldots,j=1,2,\ldots,n'+1\}}^{m'} \pi_1(1, x'_i)\rho_1(1, y'_j) \qquad (11)$$

where $I_{i1,i2,j1,j2}(p,q) = \begin{cases} 1 & \text{if } x'_{i_1} \geq y'_{j_1}, x'_{i_2} < y'_{j_2}, x'_{i_1} + y'_{j_1} < 2p, \\ 0 & \text{othervise,} \end{cases}$

$$\pi_1(q,x) = \begin{cases} \sum_{i=0}^{q-1} e^{-\delta(x+)} \dfrac{\delta(x^+)^i}{i!} - \sum_{i=0}^{q-1} e^{-\delta(x)} \dfrac{\delta(x)^i}{i!} & \text{if } x > -\infty \\ \sum_{i=0}^{q-1} e^{-\delta(-\infty)} \dfrac{\delta(-\infty)^i}{i!} & \text{if } x = -\infty \end{cases}$$

$$\pi_1(q,x_1,x_2) = \begin{cases} \left(\dfrac{\delta(x_1)^q}{q!} - \dfrac{\delta(x_1^+)^q}{q!}\right)\left(e^{-\delta(x_2^+)} - e^{-\delta(x_2)}\right) & \text{if } -\infty < x_2 < x_1 \\ \pi_1(q,x_1) - \left(\dfrac{\delta(x_1)^q}{q!} - \dfrac{\delta(x_1^+)^q}{q!}\right)e^{-\delta(x_1)} & \text{if } -\infty < x_1 = x_2 \\ \left(\dfrac{\delta(x_1)^q}{q!} - \dfrac{\delta(x_1^+)^q}{q!}\right)e^{-\delta(-\infty)} & \text{if } -\infty = x_1 < x_2 \\ 0 & \text{if } -\infty < x_1 < x_2 \end{cases}$$

$$\rho_1(q,y) = \begin{cases} \sum_{i=0}^{q-1} e^{-\sigma(y^-)} \dfrac{\sigma(y^-)^i}{i!} - \sum_{i=0}^{q-1} e^{-\sigma(y)} \dfrac{\delta(y)^i}{i!} & \text{if } y < \infty \\ \sum_{i=0}^{q-1} e^{-\sigma(-\infty)} \dfrac{\sigma(\infty)^i}{i!} & \text{if } y = \infty \end{cases}$$

$$\rho_1(q,y_1,y_2) = \begin{cases} \left(\dfrac{\sigma(y_1)^q}{q!} - \dfrac{\sigma(y_1^-)^q}{q!}\right)\left(e^{-\sigma(y_2^-)} - e^{-\delta(y_2)}\right) & \text{if } y_1 < y_2 < \infty \\ \rho_1(q,y_1) - \left(\dfrac{\sigma(y_1)^q}{q!} - \dfrac{\sigma(y_1^-)^q}{q!}\right)e^{-\sigma(y_1)} & \text{if } y_1 = y_2 < \infty \\ \left(\dfrac{\sigma(y_1)^q}{q!} - \dfrac{\sigma(y_1^-)^q}{q!}\right)e^{-\sigma(\infty)} & \text{if } y_1 < y_2 = \infty \\ 0 & \text{if } y_2 < y_1 < \infty \end{cases}$$

Proof. *The formula (10) is an application of (4). The formula (11) is an application of (5). For a detailed proof, see Šmíd (2004), Section 1.*

Remark. Note that the distribution of *(P,Q)* does not depend directly either on the number of agents or on the intensities of their arrival – it is uniquely determined only by $\delta(\bullet)$ and $\sigma(\bullet)$.

3. A Model with a Continuous Expected Demand and Supply

In the present section, let us assume that there exist functions δ and σ such that

1. $\delta(\bullet)$ is non-increasing with $\lim_{x \to \infty} \delta(x) = 0$,

2. $\sigma(\bullet)$ is non-decreasing with $\lim_{y \to -\infty} \sigma(y) = 0$,

3. $D(p) \sim \text{Po}(\delta(p))$ for each $p \in R$,

4. $D(p) - D(p')$ is independent of $D(p')$ for each $p < p'$,

5. $S(p) \sim \text{Po}(\sigma(p))$ for each $p \in R$,

6. $S(p) - S(p')$ is independent of $S(p')$ for each $p' < p$,

7. $\delta(\bullet)$ is continuous,

8. $\sigma(\bullet)$ is continuous.

Similarly to the "finite participants" model, it follows from the assumptions 3. and 5. that

$$ED(x) = \delta(x), \quad ES(y) = \sigma(y). \tag{12}$$

The joint distribution of the vector *(P,Q)* could be expressed by means of a Lebeque integral:

Theorem 3. *If δ and σ are continuous and if $q>0$ then*

$$
\begin{aligned}
P = \{Q = q, P < p\} \quad &= \int_{-\infty}^{p} e^{-\sigma(2p-y)-\delta(2p-y)} \sum_{i=0}^{q-1} \frac{\delta(2p-y)^i}{i!} \frac{\sigma(y)^{q-1}}{(q-1)!} s(y)dy \\
&- \int_{-\infty}^{p} e^{-\sigma(y)-\delta(y)} \sum_{i=0}^{q-1} \frac{\delta(y)^i}{i!} \frac{\sigma(y)^{q-1}}{(q-1)!} s(y)dy \\
&+ \int_{-\infty}^{p} e^{-\sigma(y)-\delta(y)} \sum_{i=0}^{q} \frac{\delta(y)^i}{i!} \frac{\sigma(y)^{q-1}}{q!} s(y)dy \\
&+ \int_{p}^{\infty} e^{-\sigma(y)-\delta(y)} \sum_{i=0}^{q} \frac{\delta(y)^i}{i!} \frac{\sigma(2\pi-y)^{q-1}}{q!} s(y)dy \\
&- \int_{-\infty}^{p} \left(\int_{y}^{2p-y} e^{-\sigma(z)-\delta(z)} s(z)dz \right) \frac{\delta(2p-y)^q}{q!} \frac{\sigma(y)^{q-1}}{(q-1)!} s(y)dy,
\end{aligned}
\tag{13}
$$

$$P = \{Q = 0\} = \int e^{-\sigma(y)-\delta(y)} s(y)dy + e^{-\sigma(\infty)-\delta(y)} \tag{14}$$

where $s(y) = \dfrac{\partial \sigma(y)}{\partial \sigma}$ *in all points in which the derivative exists and s(y)=0 in all the remaining points.*

Proof. *The formula (13) is an application of (4). The formula (14) is an application of (5). For a detailed proof, cf. Šmíd (2004), Section 2.*

4. Conclusion

We have determined formulas for the joint distribution of the price and the traded volume in the organized auction given two (slightly) different systems of assumptions. Both the formulas are complicated but practically computable.

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Stochastický model nelikvidního trhu s nedělitelnou komoditou

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Abstrakt

Článek popisuje nelikvidní trh, na kterém je cena určována jeho organizátorem tak, aby maximalizovala obchodované množství. V článku jsou prezentovány dva modely této situace – jeden s konečným a druhý s nekonečným počtem účastníků. V obou případech je odvozeno sdružené rozdělení tržní ceny a obchodovaného množství.

Klíčová slova: nelikvidní trh; obchodované množství; tržní cena; stochastické modely.

Stochastic model of thin market with an indivisible commodity

Abstract

In the paper, a thin market with an indivisible commodity, at which the market price is determined (by an organizer of the market) as the average price maximizing the traded volume, is modeled. Two models are presented – the first one with a finite, the second one with a possibly infinite number of participants. In both the cases, the joint distribution of the market price and the traded volume is derived.

Key words: thin market; market price; traded volume; stochastic models.

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